

L_∞ -ALGEBRAS AND DEFORMATIONS OF HOLOMORPHIC MAPS

DONATELLA IACONO

ABSTRACT. We construct the deformation functor associated with a pair of morphisms of differential graded Lie algebras, and use it to study infinitesimal deformations of holomorphic maps of compact complex manifolds. In particular, using L_∞ structures, we give an explicit description of the differential graded Lie algebra that controls this problem.

1. INTRODUCTION

The aim of this paper is to develop some algebraic tools to study infinitesimal deformations of holomorphic maps.

The modern approach to deformation theory is via differential graded Lie algebras (DGLA for short) or, in general, via L_∞ -algebras. A DGLA is a differential graded vector space with a structure of graded Lie algebra, plus some compatibility conditions between the differential and the bracket (of the Lie structure).

Moreover, using the solutions of the Maurer-Cartan equation and the gauge equivalence, we can associate with a DGLA L a deformation functor of Artin rings Def_L , i.e., a functor from the category **Art** of local Artinian \mathbb{C} -algebras (with residue field \mathbb{C}) to the category **Set** of sets, that satisfies Schlessinger's conditions (H_1) and (H_2) of [1, Theorem 2.1].

The guiding principle is the idea due to P. Deligne, V. Drinfeld, D. Quillen and M. Kontsevich (see [13]) that “in characteristic zero every deformation problem is controlled by a differential graded Lie algebra”.

In other words, we can define a DGLA L (up to quasi-isomorphism) from the geometrical data of the problem, such that the deformation functor Def_L is isomorphic to the deformation functor of Artin rings that describes the formal deformations of the geometric object [1], [7] and [19]. We point out that it is easier to study a deformation functor associated with a DGLA but, in general, it is not an easy task to find the right DGLA (up to quasi-isomorphism) associated with the problem [14].

A first example, in which the associated DGLA is well understood, is the case of deformations of complex manifolds. If X is a complex compact manifold, then its Kodaira-Spencer algebra controls the infinitesimal deformations of X (Theorem 3.4).

The next natural problem is to investigate the embedded deformations of a submanifold in a fixed manifold. Very recently, M. Manetti in [16] studies this problem using the approach via DGLA. More precisely, given a morphism of DGLAs $h : L \longrightarrow M$, he describes a general construction to define

a new deformation functor Def_h associated with h (Remark 4.4). Then, by suitably choosing L, M and h he proves the existence of an isomorphism between the functor Def_h and the functor associated with the infinitesimal deformations of a submanifold in a fixed manifold.

In this paper, we extend these techniques to study not only the deformations of an inclusion but, in general, the deformations of holomorphic maps.

These deformations were first studied from the classical point of view (no DGLA) by E. Horikawa [8] and [9], then by M. Namba [17], Z. Ran [18] and, more recently, by E. Sernesi [20].

Roughly speaking, we have a holomorphic map $f : X \longrightarrow Y$ of compact complex manifolds and we deform both the domain, the codomain and the map itself. Equivalently, we deform the graph of f in the product $X \times Y$, such that the deformation of $X \times Y$ is a product of deformations of X and Y . Let $\text{Def}(f)$ be the functor associated with the infinitesimal deformations of the holomorphic map f (Definition 5.3).

To study these deformations, the key point is the definition of the deformation functor $\text{Def}_{(h,g)}$, associated with a pair of morphisms of differential graded Lie algebras $h : L \longrightarrow M$ and $g : N \longrightarrow M$. In particular, the tangent and obstruction spaces of $\text{Def}_{(h,g)}$ are the first and second cohomology group of the suspension of the mapping cone $C_{(h,g)}$, associated with the morphism $h - g : L \oplus N \longrightarrow M$, such that $(h - g)(l, n) = h(l) - g(n)$ (Section 4.2).

By a suitable choice of the morphisms $h : L \longrightarrow M$ and $g : N \longrightarrow M$, the functor $\text{Def}_{(h,g)}$ encodes all the geometric data of the problem of infinitesimal deformations of holomorphic maps (Theorem 5.11).

Theorem (A). *Let $f : X \longrightarrow Y$ be a holomorphic map of compact complex manifolds. There exist morphisms of DGLAs $h : L \longrightarrow M$ and $g : N \longrightarrow M$ such that*

$$\text{Def}_{(h,g)} \cong \text{Def}(f).$$

Next, we look for a DGLA that controls the deformations of holomorphic maps and, for this purpose, we use L_∞ structures.

First, using path objects, we define a differential graded Lie algebra $H_{(h,g)}$, for each choice of morphisms $h : L \longrightarrow M$ and $g : N \longrightarrow M$. Then, by transferring L_∞ structures, we explicitly describe an L_∞ structure on the cone $C_{(h,g)}$ (Section 6). In particular, the functor $\text{Def}_{H_{(h,g)}}$ is isomorphic to the deformation functor $\text{Def}_{C_{(h,g)}}^\infty$ associated with this L_∞ structure on $C_{(h,g)}$ (Corollary 6.13).

Finally, we prove that the deformation functor $\text{Def}_{C_{(h,g)}}^\infty$ coincides with the deformation functor $\text{Def}_{(h,g)}$ associated with the pair (h, g) (Theorem 6.17) and so $\text{Def}_{H_{(h,g)}} \cong \text{Def}_{(h,g)}$ (Corollary 6.18).

Therefore, in particular, we give an explicit description (more than the existence) of a DGLA that controls the deformations of holomorphic maps (Theorem 6.19).

Theorem (B). *Let $f : X \longrightarrow Y$ be a holomorphic map of compact complex manifold. Then, there exists an explicit description of a DGLA $H_{(h,g)}$ such*

that

$$\mathrm{Def}_{H(h,g)} \cong \mathrm{Def}(f).$$

When we developed the techniques of this paper, we had also in mind some applications to the study of obstruction theory. However, since the number of pages grew, we decided to split the material, collecting here the general theory and leaving for the sequel [11] (in preparation) the study of obstructions and of semi-regularity maps that annihilates obstructions.

Acknowledgments. It is a pleasure for me to show my deep gratitude to the advisor of my PhD thesis Prof. Marco Manetti, an excellent helpful professor who supports and encourages me every time. I'm indebted with him for many useful discussions, advices and suggestions. Several ideas of this work are grown under his influence. This paper was written at the Mittag-Leffler Institute (Djursholm, Sweden), during my participation in the *Moduli Spaces* Program, Spring 2007. I am very grateful for the hospitality and support received. I am also grateful to the referee for improvements in the presentation of the paper.

2. NOTATION

We will work over the field \mathbb{C} of complex numbers, although most of the algebraic results are valid over an arbitrary field of characteristic zero. All vector spaces, linear maps, tensor products etc. are intended over \mathbb{C} .

Unless otherwise specified, any (complex) manifold is assumed compact and connected.

Given a manifold X , we denote by Θ_X the holomorphic tangent bundle, by $\mathcal{A}_X^{p,q}$ the sheaf of differentiable (p, q) -forms on X and by $A_X^{p,q} = \Gamma(X, \mathcal{A}_X^{p,q})$ the vector space of global sections of $\mathcal{A}_X^{p,q}$. More generally, $\mathcal{A}_X^{p,q}(\Theta_X)$ is the sheaf of differentiable (p, q) -forms on X with values in Θ_X and $A_X^{p,q}(\Theta_X) = \Gamma(X, \mathcal{A}_X^{p,q}(\Theta_X))$ is the vector space of its global sections.

Let $f : X \rightarrow Y$ be a holomorphic map of manifolds. We denote by f^* and f_* the map induced by f , i.e.,

$$f^* : A_Y^{p,q}(\Theta_Y) \rightarrow A_X^{p,q}(f^*\Theta_Y) \quad \text{and} \quad f_* : A_X^{p,q}(\Theta_X) \rightarrow A_X^{p,q}(f^*\Theta_Y).$$

3. BACKGROUND

Let $L = (\oplus_i L_i, d, [\cdot, \cdot])$ be a DGLA and $(A, m_A) \in \mathbf{Art}$, where m_A denotes the maximal ideal of A . The set of Maurer-Cartan elements with coefficients in A is defined as follows

$$\mathrm{MC}_L(A) = \{x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0\},$$

where the DGLA structure on $L \otimes m_A$ is the natural extension of the DGLA structure on L . For each $a \in L^0 \otimes m_A$, we define the gauge action $* : \exp(L^0 \otimes m_A) \times \mathrm{MC}_L(A) \rightarrow \mathrm{MC}_L(A)$ by the formula

$$e^a * x := x + \sum_{n \geq 0} \frac{[a, -]^n}{(n+1)!}([a, x] - da).$$

Given $x \in \mathrm{MC}_L(A)$, the *irrelevant stabilizer* $\mathrm{Stab}_A(x)$ of x is by definition

$$\mathrm{Stab}_A(x) = \{e^{dh+[x,h]} \mid h \in L^{-1} \otimes m_A\}.$$

The set $Stab_A(x)$ is a subgroup of $\exp(L^0 \otimes A)$, that is contained in the stabilizer of x and it satisfies the following property:

$$\forall a \in L^0 \otimes A \quad e^a Stab_A(x) e^{-a} = Stab_A(y), \quad \text{with} \quad y = e^a * x.$$

The deformation functor $\text{Def}_L : \mathbf{Art} \longrightarrow \mathbf{Set}$ associated with a DGLA L is:

$$\text{Def}_L(A) = \frac{\{x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0\}}{\exp(L^0 \otimes A)}.$$

Definition 3.1. A functor of Artin rings $F : \mathbf{Art} \longrightarrow \mathbf{Set}$ is *controlled* by a DGLA L if F is isomorphic to Def_L .

Example 3.2. Let X be a manifold. The *Kodaira-Spencer* (differential graded Lie) algebra of X is

$$KS_X = \bigoplus_i \Gamma(X, \mathcal{A}_X^{0,i}(\Theta_X)) = \bigoplus_i \mathcal{A}_X^{0,i}(\Theta_X).$$

The differential \tilde{d} is the opposite of the Dolbeault differential, whereas the bracket $[,]$ is defined in local coordinates as the $\overline{\Omega}^*$ -bilinear extension of the standard bracket on $\mathcal{A}_X^{0,0}(\Theta_X)$ ($\overline{\Omega}^* = \ker(\partial : \mathcal{A}_X^{0,*} \longrightarrow \mathcal{A}_X^{1,*})$ is the sheaf of anti-holomorphic differential forms). Explicitly, if z_1, \dots, z_n are local holomorphic coordinates on X , we have

$$\begin{aligned} \tilde{d}(f d\bar{z}_I \frac{\partial}{\partial z_i}) &= -\bar{\partial}(f) \wedge d\bar{z}_I \frac{\partial}{\partial z_i}, \\ [f \frac{\partial}{\partial z_i} d\bar{z}_I, g \frac{\partial}{\partial z_j} d\bar{z}_J] &= (f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i}) d\bar{z}_I \wedge d\bar{z}_J, \quad \forall f, g \in \mathcal{A}_X^{0,0}. \end{aligned}$$

Then, $\mathcal{A}_X^{0,*}(\Theta_X)$ is a sheaf of DGLAs.

Define the *holomorphic Lie derivative*

$$\begin{aligned} \mathbf{l} : \mathcal{A}_X^{0,*}(\Theta_X) &\longrightarrow \text{Der}^*(\mathcal{A}_X^{*,*}), \\ \mathbf{l}_a(\omega) &= \partial(a \lrcorner \omega) + (-1)^{\deg(a)} a \lrcorner \partial \omega, \end{aligned}$$

for each $a \in \mathcal{A}_X^{0,*}(\Theta_X)$ and $\omega \in \mathcal{A}_X^{*,*}$.

The DGLA sheaf morphism \mathbf{l} is injective; moreover, using \mathbf{l} , we define, for any object $(A, m_A) \in \mathbf{Art}$ and $a \in \mathcal{A}_X^{0,0}(\Theta_X) \otimes m_A$, the automorphism e^a of $\mathcal{A}_X^{0,*} \otimes A$:

$$(1) \quad e^a : \mathcal{A}_X^{0,*} \otimes A \longrightarrow \mathcal{A}_X^{0,*} \otimes A, \quad f \longmapsto e^a(f) = \sum_{n=0}^{\infty} \frac{\mathbf{l}_a^n}{n!}(f).$$

Lemma 3.3. For every local Artinian \mathbb{C} -algebra (A, m_A) , $a \in \mathcal{A}_X^{0,0}(\Theta_X) \otimes m_A$ and $x \in \text{MC}_{KS_X}(A)$ we have

$$(2) \quad e^a \circ (\bar{\partial} + \mathbf{l}_x) \circ e^{-a} = \bar{\partial} + e^a * \mathbf{l}_x : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A,$$

where $*$ is the gauge action. In particular,

$$\ker(\bar{\partial} + e^a * \mathbf{l}_x : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A) = e^a(\ker(\bar{\partial} + \mathbf{l}_x : \mathcal{A}_X^{0,0} \otimes A \longrightarrow \mathcal{A}_X^{0,1} \otimes A)).$$

Proof. See [16, Lemma 5.1] or [10, Lemma II.5.5]. □

Let $\text{Def}_X: \mathbf{Art} \rightarrow \mathbf{Set}$ be the functor of infinitesimal deformations of X , i.e.,

$$\text{Def}_X(A) = \frac{\{\text{deformations } X_A \text{ of } X \text{ over } \text{Spec}(A)\}}{\sim}.$$

Recall that a deformation X_A of X over $\text{Spec}(A)$ is nothing else than a morphism $\mathcal{O}_{X_A} \rightarrow \mathcal{O}_X$ of sheaves of A -algebras such that \mathcal{O}_{X_A} is flat over A and the induced map $\mathcal{O}_{X_A} \otimes_A \mathbb{C} \rightarrow \mathcal{O}_X$ is an isomorphism. Moreover, Def_X has $H^1(X, \Theta_X)$ and $H^2(X, \Theta_X)$ as tangent and obstruction space, respectively.

The following theorem is well known and a proof based on the theorem of Newlander-Nirenberg can be found in [2], [6] or more recently in [15]. For a proof that avoids this theorem see [10, Theorem II.7.3].

Theorem 3.4. *Let X be a manifold and KS_X its Kodaira-Spencer algebra. Then there exists an isomorphism of functors*

$$\gamma': \text{Def}_{KS_X} \longrightarrow \text{Def}_X,$$

defined in the following way: given a local Artinian \mathbb{C} -algebra (A, m_A) and a solution of the Maurer-Cartan equation $x \in A_X^{0,1}(\Theta_X) \otimes m_A$, we set

$$\mathcal{O}_{X_A}(x) = \ker(\mathcal{A}_X^{0,0} \otimes A \xrightarrow{\bar{\partial} + l_x} \mathcal{A}_X^{0,1} \otimes A),$$

and the map $\mathcal{O}_{X_A}(x) \rightarrow \mathcal{O}_X$ is induced by the projection $\mathcal{A}_X^{0,0} \otimes A \rightarrow \mathcal{A}_X^{0,0} \otimes \mathbb{C} = \mathcal{A}_X^{0,0}$.

4. DEFORMATION FUNCTOR OF A PAIR OF MORPHISMS OF DGLAS

Let $h: L \rightarrow M$ be a morphism of DGLAs. The suspension of the mapping cone of h is the complex (C_h, δ) , where $C_h^i = L^i \oplus M^{i-1}$ and $\delta(l, m) = (dl, h(l) - dm)$.

Let $h: (L, d) \rightarrow (M, d)$ and $g: (N, d) \rightarrow (M, d)$ be morphisms of DGLAs:

$$\begin{array}{ccc} & & L \\ & & \downarrow h \\ N & \xrightarrow{g} & M. \end{array}$$

The suspension of the mapping cone of the pair (h, g) is the differential graded vector space $(C_{(h,g)}, D)$, where

$$C_{(h,g)}^i = L^i \oplus N^i \oplus M^{i-1}$$

and the differential D is defined as follows

$$L^i \oplus N^i \oplus M^{i-1} \ni (l, n, m) \xrightarrow{D} (dl, dn, -dm - g(n) + h(l)) \in L^{i+1} \oplus N^{i+1} \oplus M^i.$$

The projection $C_{(h,g)} \rightarrow L \oplus N$ is a morphism of complexes and so there exists the following exact sequence

$$0 \longrightarrow (M^{-1}, -d) \longrightarrow (C_{(h,g)}, D) \longrightarrow (L \oplus N, d) \longrightarrow 0$$

that induces

$$(3) \quad \cdots \longrightarrow H^i(C_{(h,g)}) \longrightarrow H^i(L \oplus N) \longrightarrow H^i(M) \longrightarrow H^{i+1}(C_{(h,g)}) \longrightarrow \cdots$$

Note that, in general, we can not define any bracket on the cone $C_{(h,g)}$, such that $C_{(h,g)}$ is a DGLA and the projection $C_{(h,g)} \longrightarrow L \oplus N$ is a morphism of DGLAs. In Section 6, we will define an L_∞ structure on $C_{(h,g)}$.

Lemma 4.1. *Let $g : N \longrightarrow M$ and $h : L \longrightarrow M$ be morphisms of complexes with h injective, i.e., there exists the exact sequence of complexes*

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{h} & M & \xrightarrow{\pi} & \text{coker}(h) \longrightarrow 0 \\ & & & & \uparrow g & & \\ & & & & N & & \end{array}$$

Then, $(C_{(h,g)}, D)$ is quasi-isomorphic to $(C_{\pi \circ g}, \delta)$.

Proof. Let $\gamma : C_{(h,g)} \longrightarrow C_{\pi \circ g}$ be defined as

$$C_{(h,g)}^i \ni (l, n, m) \xrightarrow{\gamma} (-n, \pi(m)) \in C_{\pi \circ g}^i.$$

Then, a straightforward computation shows that γ is a quasi-isomorphism. \square

4.1. The functors $\text{MC}_{(h,g)}$ and $\text{Def}_{(h,g)}$. The *Maurer-Cartan functor associated with the pair (h, g)* is defined as follows

$$\text{MC}_{(h,g)} : \mathbf{Art} \longrightarrow \mathbf{Set},$$

$$\begin{aligned} \text{MC}_{(h,g)}(A) &= \{(x, y, e^p) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid \\ &\quad dx + \frac{1}{2}[x, x] = 0, \quad dy + \frac{1}{2}[y, y] = 0, \quad g(y) = e^p * h(x)\}. \end{aligned}$$

We note that $\text{MC}_{(h,g)}$ is a *homogeneous* functor, i.e., for each $B \longrightarrow A$ and $C \longrightarrow A$ in \mathbf{Art} , $\text{MC}_{(h,g)}(B \times_A C) \cong \text{MC}_{(h,g)}(B) \times_{\text{MC}_{(h,g)}(A)} \text{MC}_{(h,g)}(C)$.

Remark 4.2. In [16, Section 2], M. Manetti introduced the functor $\text{MC}_h : \mathbf{Art} \longrightarrow \mathbf{Set}$, associated with a morphism $h : L \longrightarrow M$ of DGLAs, by defining, for each $(A, m_A) \in \mathbf{Art}$,

$$\text{MC}_h(A) =$$

$$\{(x, e^p) \in (L^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid dx + \frac{1}{2}[x, x] = 0, \quad e^p * h(x) = 0\}.$$

Therefore, if we take $N = 0$ and $g = 0$, the new functor $\text{MC}_{(h,g)}$ reduces to the old one MC_h . By choosing $M = N = 0$ and $h = g = 0$, $\text{MC}_{(h,g)}$ reduces to the Maurer-Cartan functor MC_L associated with the DGLA L .

Next, we consider on $\text{MC}_{(h,g)}(A)$ the following relation \sim :

$$(x_1, y_1, e^{p_1}) \sim (x_2, y_2, e^{p_2})$$

if and only if there exist $a \in L^0 \otimes m_A$, $b \in N^0 \otimes m_A$ and $c \in M^{-1} \otimes m_A$ such that

$$x_2 = e^a * x_1, \quad y_2 = e^b * y_1$$

and

$$e^{p_2} = e^{g(b)} e^T e^{p_1} e^{-h(a)}, \quad \text{with} \quad T = dc + [g(y_1), c].$$

By definition of the irrelevant stabilizer, we note that $e^T \in \text{Stab}_A(g(y_1))$. An easy computation shows that \sim is a well defined equivalence relation [10, Lemma III.2.23]. Then, it makes sense to consider the following functor.

Definition 4.3. The *deformation functor associated with a pair (h, g)* of morphisms of differential graded Lie algebras is:

$$\text{Def}_{(h,g)} : \mathbf{Art} \longrightarrow \mathbf{Set},$$

$$\text{Def}_{(h,g)}(A) = \frac{\text{MC}_{(h,g)}(A)}{\sim}.$$

Remark 4.4. In [16, Section 2], M. Manetti defined the functor Def_h associated with a morphism $h : L \longrightarrow M$ of DGLAs:

$$\text{Def}_h : \mathbf{Art} \longrightarrow \mathbf{Set},$$

$$\text{Def}_h(A) = \frac{\text{MC}_h(A)}{\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)},$$

where the gauge action of $\exp(L^0 \otimes m_A) \times \exp(dM^{-1} \otimes m_A)$ is given by the formula

$$(e^a, e^{dm}) * (x, e^p) = (e^a * x, e^{dm} e^p e^{-h(a)}), \quad \forall a \in L^0 \otimes m_A, m \in M^{-1} \otimes m_A.$$

Therefore, if we take $N = 0$ and $g = 0$, the new functor $\text{Def}_{(h,g)}$ reduces to the old one Def_h .

By choosing $N = M = 0$ and $h = g = 0$, $\text{Def}_{(h,g)}$ reduces to the Maurer-Cartan functor Def_L associated with the DGLA L .

Remark 4.5. Consider the functor $\text{Def}_{(h,g)}$. Then the projection ϱ on the second factor:

$$\varrho : \text{Def}_{(h,g)} \longrightarrow \text{Def}_N,$$

$$\text{Def}_{(h,g)}(A) \ni (x, y, e^p) \xrightarrow{\varrho} y \in \text{Def}_N(A)$$

is a morphism of deformation functors.

Remark 4.6. If the morphism h is injective, then for each $(A, m_A) \in \mathbf{Art}$ the functor $\text{MC}_{(h,g)}$ has the following form:

$$\text{MC}_{(h,g)}(A) = \{(x, e^p) \in (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) |$$

$$dx + \frac{1}{2}[x, x] = 0, \quad e^{-p} * g(x) \in L^1 \otimes m_A\}.$$

If M is also concentrated in non negative degrees, then the gauge equivalence is given by

$$(x, e^p) \sim (e^b * x, e^{g(b)} e^p e^a), \quad \text{with } a \in L^0 \otimes m_A \text{ and } b \in N^0 \otimes m_A.$$

4.2. Tangent and obstruction spaces of $\mathrm{MC}_{(h,g)}$ and $\mathrm{Def}_{(h,g)}$. By definition, the tangent space of a functor of Artin rings F is $F(\mathbb{C}[\varepsilon])$, where $\varepsilon^2 = 0$. Therefore,

$$\begin{aligned} \mathrm{MC}_{(h,g)}(\mathbb{C}[\varepsilon]) &= \\ &= \{(x, y, e^p) \in (L^1 \otimes \mathbb{C}\varepsilon) \times (N^1 \otimes \mathbb{C}\varepsilon) \times \exp(M^0 \otimes \mathbb{C}\varepsilon) \mid \\ &\quad dx = dy = 0, h(x) - g(y) - dp = 0\} \\ &\cong \{(x, y, p) \in L^1 \times N^1 \times M^0 \mid dx = dy = 0, -dp - g(y) + h(x) = 0\} = \\ &\quad \ker(D : C_{(h,g)}^1 \longrightarrow C_{(h,g)}^2), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Def}_{(h,g)}(\mathbb{C}[\varepsilon]) &\cong \\ &\frac{\{(x, y, p) \in L^1 \times N^1 \times M^0 \mid dx = dy = 0, g(y) = h(x) - dp\}}{\{(-da, -db, dc + g(b) - h(a)) \mid a \in L^0, b \in N^0, c \in M^{-1}\}} \\ &\cong H^1(C_{(h,g)}). \end{aligned}$$

The obstruction space of $\mathrm{Def}_{(h,g)}$, is naturally contained in $H^2(C_{(h,g)})$. Indeed, let

$$0 \longrightarrow J \longrightarrow \tilde{A} \xrightarrow{\alpha} A \longrightarrow 0$$

be a small extension and $(x, y, e^p) \in \mathrm{MC}_{(h,g)}(A)$.

Since α is surjective, there exist $\tilde{x} \in L^1 \otimes m_{\tilde{A}}$ that lifts x , $\tilde{y} \in N^1 \otimes m_{\tilde{A}}$ that lifts y , and $q \in M^0 \otimes m_{\tilde{A}}$ that lifts p . Let

$$l = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes m_{\tilde{A}}$$

and

$$k = d\tilde{y} + \frac{1}{2}[\tilde{y}, \tilde{y}] \in N^2 \otimes m_{\tilde{A}}.$$

It is easy to see that $\alpha(l) = \alpha(k) = dl = dk = 0$; then $l \in H^2(L) \otimes J$ and $k \in H^2(N) \otimes J$.

Let $r = -g(\tilde{y}) + e^q * h(\tilde{x}) \in M^1 \otimes m_{\tilde{A}}$; thus, $\alpha(r) = 0$ or, equivalently, $r \in M^1 \otimes J$. It can be proved that $-dr - g(k) + h(l) = 0$ and so $(l, k, r) \in Z^2(C_{(h,g)} \otimes J)$. Let $[(l, k, r)]$ be the class in $H^2(C_{(h,g)} \otimes J)$. This class does not depend on the choice of the liftings and it vanishes if and only if there exists a lifting of $(x, y, e^p) \in \mathrm{MC}_{(h,g)}(A)$ in $\mathrm{MC}_{(h,g)}(\tilde{A})$ ([10, Lemma III.1.19]).

5. DEFORMATIONS OF HOLOMORPHIC MAPS

Definition 5.1. Let $f : X \longrightarrow Y$ be a holomorphic map of manifolds and $A \in \mathbf{Art}$. An *infinitesimal deformation of f over $\mathrm{Spec}(A)$* is a commutative diagram of complex spaces

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ & \searrow \pi \quad \swarrow \mu & \\ & \mathrm{Spec}(A), & \end{array}$$

where $(X_A, \pi, \mathrm{Spec}(A))$ and $(Y_A, \mu, \mathrm{Spec}(A))$ are infinitesimal deformations of X and Y , respectively, and \mathcal{F} is a holomorphic map that restricted to the fibers over the closed point of $\mathrm{Spec}(A)$ coincides with f . If $A = \mathbb{C}[\varepsilon]$ we have a *first order deformation* of f .

Definition 5.2. Let

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ & \searrow \pi & \swarrow \mu \\ & \text{Spec}(A) & \end{array} \quad \text{and} \quad \begin{array}{ccc} X'_A & \xrightarrow{\mathcal{F}'} & Y'_A \\ & \searrow \pi' & \swarrow \mu' \\ & \text{Spec}(A) & \end{array}$$

be two infinitesimal deformations of f over $\text{Spec}(A)$. They are *isomorphic* if there exist bi-holomorphic maps $\phi : X_A \rightarrow X'_A$ and $\psi : Y_A \rightarrow Y'_A$ (that are equivalences of infinitesimal deformations of X and Y , respectively) such that the following diagram is commutative:

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ \phi \downarrow & & \downarrow \psi \\ X'_A & \xrightarrow{\mathcal{F}'} & Y'_A \end{array}$$

Definition 5.3. The *functor of infinitesimal deformations* of a holomorphic map $f : X \rightarrow Y$ is

$$\text{Def}(f) : \mathbf{Art} \rightarrow \mathbf{Set},$$

$$A \mapsto \text{Def}(f)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations} \\ \text{of } f \text{ over } \text{Spec}(A) \end{array} \right\}.$$

Remark 5.4. Let Γ be the graph of f in the product $X \times Y$. The infinitesimal deformations of f can be interpreted as infinitesimal deformations Γ_A of Γ in the product $X \times Y$, such that the induced deformations $(X \times Y)_A$ of $X \times Y$ are products of infinitesimal deformations of X and of Y . Since not all the deformations of a product are products of deformations ([12, pag. 436]), we are not just considering the deformations of the graph in the product. Moreover, with this interpretation, two infinitesimal deformations $\Gamma_A \subset (X \times Y)_A$ and $\Gamma'_A \subset (X \times Y)'_A$ are equivalent if there exists an isomorphism $\phi : (X \times Y)_A \rightarrow (X \times Y)'_A$ of infinitesimal deformations of $X \times Y$ such that $\phi(\Gamma_A) = \Gamma'_A$.

Let $(B^\cdot, D_{\bar{\partial}})$ be the complex with

$$B^p = A_X^{(0,p)}(\Theta_X) \oplus A_Y^{(0,p)}(\Theta_Y) \oplus A_X^{(0,p-1)}(f^*\Theta_Y)$$

and

$$D_{\bar{\partial}} : B^p \rightarrow B^{p+1}, \quad (x, y, z) \mapsto (\bar{\partial}x, \bar{\partial}y, \bar{\partial}z + (-1)^p(f_*x - f^*y)).$$

Theorem 5.5 (E. Horikawa). $H^1(B^\cdot)$ is in one-to-one correspondence with the first order deformations of $f : X \rightarrow Y$.

The obstruction space of the functor $\text{Def}(f)$ is naturally contained in $H^2(B^\cdot)$.

Proof. See [17, Section 3.6]. □

Remark 5.6. Consider a first order deformation f_ε of f : in particular, we are considering first order deformations X_ε and Y_ε , of X and Y , respectively.

Then, we associate with X_ε a class $x \in H^1(X, \Theta_X)$ and with Y_ε a class $y \in H^1(Y, \Theta_Y)$. Therefore, the class in $H^1(B)$ associated with f_ε is $[(x, y, z)]$, with $z \in A_X^{(0,0)}(f^*\Theta_Y)$ such that $\bar{\partial}z = f_*x - f^*y$.

Analogously, let $0 \rightarrow J \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ be a small extension and \mathcal{F}_A an infinitesimal deformation of f over $\text{Spec}(A)$. If $h \in H^2(X, \Theta_X)$ and $k \in H^2(Y, \Theta_Y)$ are the obstruction classes associated with X_A and Y_A , respectively, then the obstruction class in $H^2(B)$ associated with \mathcal{F}_A is $[(h, k, r)]$, with $r \in A_X^{(0,1)}(f^*\Theta_Y)$ such that $\bar{\partial}r = -(f_*h - f^*k)$.

Let $Z := X \times Y$ be the product of X and Y and $p : Z \rightarrow X$ and $q : Z \rightarrow Y$ the natural projections. Defining the morphism

$$F : X \rightarrow \Gamma \subseteq Z,$$

$$x \mapsto (x, f(x)),$$

we get the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{F} & Z & & \\ & \searrow f & \downarrow p & \searrow q & \\ & & X & & Y \\ & \swarrow id & & \swarrow & \end{array}$$

In particular, $F^* \circ p^* = id$ and $F^* \circ q^* = f^*$. Since $\Theta_Z = p^*\Theta_X \oplus q^*\Theta_Y$, it follows that $F^*(\Theta_Z) = \Theta_X \oplus f^*\Theta_Y$. Define the morphism $\gamma : \Theta_Z \rightarrow f^*\Theta_Y$ as the product

$$\gamma : \Theta_Z \xrightarrow{F^*} \Theta_X \oplus f^*\Theta_Y \xrightarrow{(f^*, -id)} f^*\Theta_Y;$$

moreover, let π be the following surjective morphism:

$$A_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} A_X^{0,*}(f^*\Theta_Y) \rightarrow 0,$$

$$\pi(\omega u) = F^*(\omega)\gamma(u), \quad \forall \omega \in A_Z^{0,*}, u \in \Theta_Z.$$

Since each $u \in \Theta_Z$ can be written as $u = p^*v_1 + q^*v_2$, for some $v_1 \in \Theta_X$ and $v_2 \in \Theta_Y$, we also have

$$\pi(\omega u) = F^*(\omega)(f_*(v_1) - f^*(v_2)).$$

Since $F^*\bar{\partial} = \bar{\partial}F^*$, π is a morphism of complexes.

Let \mathcal{L} be the kernel of π :

$$(4) \quad 0 \rightarrow \mathcal{L} \xrightarrow{h} \mathcal{A}_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} \mathcal{A}_X^{0,*}(f^*\Theta_Y) \rightarrow 0$$

and $h : \mathcal{L} \rightarrow \mathcal{A}_Z^{0,*}(\Theta_Z)$ the inclusion.

Since there is a canonical isomorphism between the normal bundle $N_{\Gamma|Z}$ of Γ in Z and the pull-back f^*T_Y , (4) reduces to

$$0 \rightarrow \mathcal{L} \xrightarrow{h} \mathcal{A}_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} \mathcal{A}_\Gamma^{0,*}(N_{\Gamma|Z}) \rightarrow 0.$$

Let $i : \Gamma \rightarrow Z$ be the inclusion and $i^* : \mathcal{A}_Z^{0,*} \rightarrow \mathcal{A}_\Gamma^{0,*}$ the induced map. Suppose that z_1, \dots, z_n are holomorphic coordinates on Z such that

$Z \supset \Gamma = \{z_{t+1} = \cdots = z_n = 0\}$. Then, $\omega = \sum_{j=1}^n \omega_j \frac{\partial}{\partial z_j} \in \mathcal{A}_Z^{0,*}(\Theta_Z)$ lies in \mathcal{L} if and only if $\omega_j \in \ker i^*$ for $j \geq t+1$. In particular, \mathcal{L}^0 is the sheaf of differentiable vector fields on Z that are tangent to Γ .

Lemma 5.7. *\mathcal{L} is a sheaf of differential graded Lie subalgebras of $\mathcal{A}_Z^{0,*}(\Theta_Z)$ such that $\mathbf{l}_a(\ker i^*) \subset \ker i^*$ if and only if $a \in \mathcal{L} \subset \mathcal{A}_Z^{0,*}(\Theta_Z)$. Moreover, consider the automorphism e^a of $\mathcal{A}_Z^{0,*} \otimes A$ defined in (1): if $a \in \mathcal{L}^0 \otimes m_A$ then $e^a(\ker(i^*) \otimes A) = \ker(i^*) \otimes A$.*

Proof. See [16, Section 5]. It is an easy calculation in local holomorphic coordinates. \square

Let L be the differential graded Lie algebra of global sections of \mathcal{L} .

Let M be the Kodaira-Spencer algebra of the product Z , i.e. $M = KS_Z$, and $h : L \rightarrow M$ be the inclusion.

Let $N = KS_X \times KS_Y$ be the product of the Kodaira-Spencer algebras of X and of Y and $g = p^* + q^* : KS_X \times KS_Y \rightarrow KS_Z$, i.e., $g(n_1, n_2) = p^*n_1 + q^*n_2$ (for $n = (n_1, n_2)$ we also use the notation $g(n)$).

Therefore, we get the diagram

$$(5) \quad \begin{array}{ccc} & & L \\ & & \downarrow h \\ N = KS_X \times KS_Y & \xrightarrow{g=(p^*, q^*)} & M = KS_Z \end{array}$$

Remark 5.8. Given morphisms of DGLAs $h : L \rightarrow KS_Z$ and $g : KS_X \times KS_Y \rightarrow KS_Z$, we can consider the complex $(C_{(h,g)}^\bullet, D)$, with $C_{(h,g)}^i = L^i \oplus KS_X^i \oplus KS_Y^i \oplus KS_Z^{i-1}$ and differential is given by $D(l, n_1, n_2, m) = (-\bar{\partial}l, -\bar{\partial}n_1, -\bar{\partial}n_2, \bar{\partial}m - p^*n_1 - q^*n_2 + h(l))$.

Using the morphism $\pi : KS_{X \times Y} \rightarrow A_X^{0,*}(f^*\Theta_Y)$, we can define a morphism

$$\beta : (C_{(h,g)}^\bullet, D) \rightarrow (B^\bullet, D_{\bar{\partial}}),$$

$$\beta(l, n_1, n_2, m) = ((-1)^i n_1, (-1)^i n_2, -\pi(m)) \quad \forall (l, n_1, n_2, m) \in C_{(h,g)}^i.$$

Lemma 5.9. $\beta : (C_{(h,g)}^\bullet, D) \rightarrow (B^\bullet, D_{\bar{\partial}})$ is a morphism of complexes which is a quasi-isomorphism.

Proof. It follows from an easy computation. \square

Let us consider the functor $\text{Def}_{(h,g)}$ associated with diagram (5). Since h is injective and M is concentrated in non negative degrees, by Remark 4.6, for each $(A, m_A) \in \mathbf{Art}$, we have

$$\text{Def}_{(h,g)}(A) = \{(n, e^m) \in (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) |$$

$$dn + \frac{1}{2}[n, n] = 0, e^{-m} * g(n) \in L^1 \otimes m_A\} / \sim,$$

where $(x, e^p) \sim (e^b * x, e^{g(b)} e^p e^a)$, with $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$.

Remark 5.10. Let $(n, e^m) \in \text{Def}_{(h,g)}$. In particular, $n = (n_1, n_2)$ satisfies the Maurer-Cartan equation and so $n_1 \in \text{MC}_{KS_X}$ and $n_2 \in \text{MC}_{KS_Y}$. Therefore, there are associated with n infinitesimal deformations X_A of X (induced by n_1) and Y_A of Y (induced by n_2). Moreover, since $g(n)$ satisfies the Maurer-Cartan equation in $M = KS_Z$, it defines an infinitesimal deformation Z_A of Z . By construction, the deformation Z_A is the product of the deformations X_A and Y_A .

Consider an infinitesimal deformation of the holomorphic map f over $\text{Spec}(A)$ as an infinitesimal deformation Γ_A of Γ over $\text{Spec}(A)$ and Z_A of Z over $\text{Spec}(A)$, with Z_A product of deformations of X and Y over $\text{Spec}(A)$.

By applying Remark 5.10 and Theorem 3.4, the condition on the deformation Z_A is equivalent to requiring $\mathcal{O}_{Z_A} = \mathcal{O}_{Z_A}(g(n))$, for some Maurer-Cartan element $n \in KS_X \times KS_Y$. Let $i^* : \mathcal{A}_Z^{0,*} \longrightarrow \mathcal{A}_\Gamma^{0,*}$ be the restriction morphism and let $\mathcal{I} = \ker i^* \cap \mathcal{O}_Z$ be the holomorphic ideal sheaf of the graph Γ of f in Z . The deformations Γ_A of the graph Γ correspond to infinitesimal deformations $\mathcal{I}_A \subset \mathcal{O}_{Z_A}$ of \mathcal{I} over $\text{Spec}(A)$, with \mathcal{I}_A ideal sheaves of \mathcal{O}_{Z_A} , flat over A and such that $\mathcal{I}_A \otimes_A \mathbb{C} \cong \mathcal{I}$.

In conclusion, to give an infinitesimal deformation of f over $\text{Spec}(A)$ (an element in $\text{Def}(f)(A)$), it is sufficient to give an ideal sheaf $\mathcal{I}_A \subset \mathcal{O}_{Z_A}(g(n))$ (for some $n \in \text{MC}_{KS_X \times KS_Y}$) with \mathcal{I}_A A -flat and $\mathcal{I}_A \otimes_A \mathbb{C} \cong \mathcal{I}$.

Theorem 5.11. *Let h, g and i^* be as above. Then, there exists an isomorphism of functors*

$$\gamma : \text{Def}_{(h,g)} \longrightarrow \text{Def}(f).$$

Given a local Artinian \mathbb{C} -algebra A and an element $(n, e^m) \in \text{MC}_{(h,g)}(A)$, we define a deformation of f over $\text{Spec}(A)$ as a deformation $\mathcal{I}_A(n, e^m)$ of the holomorphic ideal sheaf of the graph of f in the following way

$$\begin{aligned} \gamma(n, e^m) = \mathcal{I}_A(n, e^m) &:= (\ker(\mathcal{A}_Z^{0,0} \otimes A \xrightarrow{\bar{\partial} + \mathbf{L}_{g(n)}} \mathcal{A}_Z^{0,1} \otimes A) \cap e^m(\ker i^* \otimes A)) \\ &= \mathcal{O}_{Z_A}(g(n)) \cap e^m(\ker i^* \otimes A), \end{aligned}$$

where $\mathcal{O}_{Z_A}(g(n))$ is the infinitesimal deformation of Z , given by Theorem 3.4, that corresponds to $g(n) \in \text{MC}_{KS_X \times Y}$.

Proof. For each $(n, e^m) \in \text{MC}_{(h,g)}(A)$ we have defined

$$\mathcal{I}_A(n, e^m) = \mathcal{O}_{Z_A}(g(n)) \cap e^m(\ker i^* \otimes A).$$

First of all, we verify that this sheaf $\mathcal{I}_A(n, e^m) \subset \mathcal{O}_{Z_A}(g(n))$ defines an infinitesimal deformation of f ; therefore, we need to prove that it is flat over A and $\mathcal{I}_A(n, e^m) \otimes_A \mathbb{C} \cong \mathcal{I}$. It is equivalent to verify these properties for $e^{-m}\mathcal{I}_A(n, e^m)$. Applying Lemma 3.3, yields

$$e^{-m}(\mathcal{O}_{Z_A}(g(n))) = \ker(\bar{\partial} + e^{-m} * g(n) : \mathcal{A}_Z^{0,0} \otimes A \longrightarrow \mathcal{A}_Z^{0,1} \otimes A)$$

and also

$$\begin{aligned} e^{-m}\mathcal{I}_A(n, e^m) &= e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A) = \\ &= \ker(\bar{\partial} + e^{-m} * g(n)) \cap (\ker i^* \otimes A). \end{aligned}$$

Since flatness is a local property, we can assume that Z is a Stein manifold. Then $H^1(Z, \Theta_Z) = 0$ and $H^0(Z, \Theta_Z) \rightarrow H^0(Z, N_{\Gamma|Z})$ is surjective. Since the following sequence is exact

$$\cdots \rightarrow H^0(Z, \Theta_Z) \rightarrow H^0(Z, N_{\Gamma|Z}) \rightarrow H^1(Z, L) \rightarrow H^1(Z, \Theta_Z) \rightarrow \cdots,$$

we conclude that $H^1(L) = 0$ or, equivalently, that the tangent space of the functor Def_L is trivial. Therefore, Def_L is the trivial functor.

This implies the existence of $\nu \in L^0 \otimes m_A$ such that $e^{-m} * g(n) = e^\nu * 0$ (by hypothesis, $e^{-m} * g(n)$ is a solution of the Maurer-Cartan equation in L). Moreover, we recall that $e^a(\ker i^* \otimes A) = \ker i^* \otimes A$, for each $a \in L^0 \otimes m_A$.

Therefore,

$$\begin{aligned} e^{-m} \mathcal{I}_A(n, e^m) &= \ker(\bar{\partial} + e^\nu * 0) \cap (\ker i^* \otimes A) = \mathcal{O}_{Z_A}(e^\nu * 0) \cap (\ker i^* \otimes A) \\ &= e^\nu(\mathcal{O}_{Z_A}(0)) \cap e^\nu(\ker i^* \otimes A) = e^\nu(\mathcal{I} \otimes A). \end{aligned}$$

Thus, $\mathcal{I}_A(n, e^m)$ defines a deformation of f and the morphism

$$\gamma : \text{MC}_{(h,g)} \rightarrow \text{Def}(f)$$

is well defined, such that

$$\begin{aligned} \gamma(A) : \text{MC}_{(h,g)}(A) &\rightarrow \text{Def}(f)(A) \\ (n, e^m) &\mapsto \gamma(n, e^m) = \mathcal{I}_A(n, e^m). \end{aligned}$$

Moreover, γ is well defined on $\text{Def}_{(h,g)}(A) = \text{MC}_{(h,g)}(A)/\text{gauge}$. Actually, for each $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$, we have

$$\begin{aligned} \gamma(e^b * n, e^{g(b)} e^m e^a) &= \mathcal{O}_{Z_A}(e^{g(b)} * g(n)) \cap e^{g(b)} e^m e^a(\ker i^* \otimes A) = \\ &= e^{g(b)} \mathcal{O}_{Z_A}(g(n)) \cap e^{g(b)} e^m(\ker i^* \otimes A) = e^{g(b)} \gamma(n, e^m). \end{aligned}$$

This implies that the deformations $\gamma(n, e^m)$ and $\gamma(e^b * n, e^{g(b)} e^m e^a)$ are isomorphic (Remark 5.4).

In conclusion, $\gamma : \text{Def}_{(h,g)} \rightarrow \text{Def}(f)$ is a well defined natural transformation of functors.

In order to prove that γ is an isomorphism it is sufficient to prove that

- i) γ is injective;
- ii) γ induces a bijective map on the tangent spaces;
- iii) γ induces an injective map on the obstruction spaces.

i) γ is injective. Suppose that $\gamma(n, e^m) = \gamma(r, e^s)$, then we want to prove that (n, e^m) is gauge equivalent to (r, e^s) , i.e., there exist $a \in L^0 \otimes m_A$ and $b \in N^0 \otimes m_A$ such that $e^b * r = n$ and $e^{g(b)} e^s e^a = e^m$.

By hypothesis, $\gamma(n, e^m)$ and $\gamma(r, e^s)$ are isomorphic deformations; then, in particular, the deformations induced on Z are isomorphic. This implies that there exists $b \in N^0 \otimes m_A$ such that $e^b * r = n$ and $e^{g(b)}(\mathcal{O}_{Z_A}(g(r))) = \mathcal{O}_{Z_A}(g(n))$. Up to substituting (r, e^s) with its equivalent $(e^b * r, e^{g(b)} e^s)$, we can assume to be in the following situation

$$\mathcal{O}_{Z_A}(g(n)) \cap e^m(\ker i^* \otimes A) = \mathcal{O}_{Z_A}(g(n)) \cap e^{m'}(\ker i^* \otimes A).$$

Let $e^a = e^{-m'} e^m$, then

$$e^a(e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A)) = e^{-m'}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A).$$

In particular, $e^a(e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A)) \subseteq \ker i^* \otimes A$.

Next, we prove, by induction, that $a \in L^0 \otimes m_A$ (thus $e^m = e^{m'}e^a = e^{g(b)}e^se^a$).

Let z_1, \dots, z_n be holomorphic coordinates on Z such that $Z \supset \Gamma = \{z_{t+1} = \dots = z_n = 0\}$. Consider the projection on the residue field

$$e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A) \longrightarrow \mathcal{O}_Z \cap \ker i^*.$$

Then, $z_i \in \ker i^* \cap \mathcal{O}_Z$, for $i > t$. Since $e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A)$ is flat over A , we can lift z_i to $\tilde{z}_i = z_i + \varphi_i \in e^{-m}(\mathcal{O}_{Z_A}(g(n))) \cap (\ker i^* \otimes A)$, with $\varphi_i \in \ker i^* \otimes m_A$. By hypothesis,

$$(6) \quad e^a(\tilde{z}_i) = e^a(z_i) + e^a(\varphi_i) \in \ker i^* \otimes A.$$

By Lemma 5.7, in order to prove that $a \in L^0 \otimes m_A$ it is sufficient to verify that $e^a(z_i) \in \ker i^* \otimes A$ and so, by (6), that $e^a(\varphi_i) \in \ker i^* \otimes A$.

If $A = \mathbb{C}[\varepsilon]$, then $\varphi_i \in \ker i^* \otimes \mathbb{C}\varepsilon$ and $a \in \mathcal{A}_Z^{0,0} \otimes \mathbb{C}\varepsilon$; this implies $e^a(\varphi_i) = \varphi_i \in \ker i^* \otimes \mathbb{C}\varepsilon$.

Next, let $0 \longrightarrow J \longrightarrow \tilde{A} \xrightarrow{\alpha} A \longrightarrow 0$ be a small extension. By hypothesis, $\alpha(a) \in L^0 \otimes m_A$, that is, $\alpha(a) = \sum_{j=1}^n \bar{a}_j \frac{\partial}{\partial z_j}$ with $\bar{a}_j \in \ker i^* \otimes m_A$ for $j > t$.

Let a'_j be liftings of \bar{a}_j . Then $a'_j \in \ker i^* \otimes m_{\tilde{A}}$, for $j > t$, $a' = \sum_{j=1}^n a'_j \frac{\partial}{\partial z_j} \in L^0 \otimes m_{\tilde{A}}$ and $e^{a'}(\varphi_i) \in \ker i^* \otimes m_{\tilde{A}}$. Since $\alpha(a) = \alpha(a')$, then $a = a' + j$ with $j \in M^0 \otimes J$. This implies that $e^a(\varphi_i) = e^{a'+j}(\varphi_i) = e^{a'}(\varphi_i) \in \ker i^* \otimes m_{\tilde{A}}$.

As to *ii*) and *iii*), a straightforward computation shows that the maps induced by γ on tangent and obstruction spaces are the isomorphisms induced by β of Lemma 5.9. □

Remark 5.12. Consider the diagram

$$\begin{array}{ccc} & & L \\ & & \downarrow h \\ KS_X \times KS_Y & \xrightarrow{g} & KS_{X \times Y} \\ & \searrow \pi \circ g & \searrow \pi \\ & & A_X^{0,*}(f^*T_Y). \end{array}$$

Since h is injective, Lemma 4.1 implies the existence of a quasi-isomorphism of complexes $(C_{(h,g)}, D)$ and $(C_{\pi \circ g}, \check{\delta})$.

Then, we get the following exact sequence

$$(7) \quad \begin{aligned} \cdots \longrightarrow H^1(C_{\pi \circ g}) &\xrightarrow{\varrho^1} H^1(X, \Theta_X) \oplus H^1(Y, \Theta_Y) \longrightarrow H^1(X, f^*\Theta_Y) \longrightarrow \\ &\longrightarrow H^2(C_{\pi \circ g}) \xrightarrow{\varrho^2} H^2(X, \Theta_X) \oplus H^2(Y, \Theta_Y) \longrightarrow H^2(X, f^*\Theta_Y) \longrightarrow \cdots, \end{aligned}$$

where ϱ^1 and ϱ^2 are the projections on the second factor and they are induced by the projection morphism $\varrho : \text{Def}(f) \longrightarrow \text{Def}_{KS_X \times KS_Y}$ (see Remark 4.5).

In particular, $\varrho : \text{Def}(f) \longrightarrow \text{Def}_{KS_X \times KS_Y}$ associates with an infinitesimal deformation of f the induced infinitesimal deformations of X and Y .

Therefore, ϱ^1 associates with a first order deformation of f the induced first order deformations of X and Y and ϱ^2 is a morphism of obstruction theories: the obstruction to deform f is mapped to the induced obstructions to deform X and Y (see also Remark 5.10).

6. L_∞ STRUCTURE ON THE CONE $C_{(h,g)}$

In this section we explicitly describe an L_∞ structure on the cone $C_{(h,g)}$, associated with the pair of morphisms of DGLAs $h : L \rightarrow M$ and $g : N \rightarrow M$. In particular, we prove that the deformation functor $\text{Def}_{(h,g)}$ coincides with the functor $\text{Def}_{C_{(h,g)}}^\infty$, associated with this L_∞ structure on $C_{(h,g)}$ (Theorem 6.17). Finally, we show the existence of a DGLA $H_{(h,g)}$ that controls the deformations of holomorphic maps (Corollary 6.18).

First of all, we briefly recall the definition of an L_∞ structures on a graded vector space V . For a complete description of such structures, see, for example, [4], [5], [13] or [15, Chapter IX].

We denote by $V[1]$ the complex V with degrees shifted by 1. More precisely, for $\mathbb{C}[1]$ we have

$$\mathbb{C}[1]^i = \begin{cases} \mathbb{C} & \text{if } i+1=0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $V[1] = \mathbb{C}[1] \otimes V$, which implies $V[1]^i = V^{i+1}$.

Let $\epsilon(\sigma; v_1, \dots, v_n)$ be the Koszul sign. We denote by $\odot^n V$ the space of co-invariant elements for the action of Σ_n on $\otimes^n V$ given by

$$\sigma(v_1 \otimes \dots \otimes v_n) = \epsilon(\sigma; v_1, \dots, v_n) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

When (v_1, \dots, v_n) are clear by the context we simply write $\epsilon(\sigma)$ instead of $\epsilon(\sigma; v_1, \dots, v_n)$.

Definition 6.1. The set of *unshuffles* of type (p, q) is the subset $S(p, q)$ of Σ_n of permutations σ , such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$.

Definition 6.2. An L_∞ structure on a graded vector space V is a system $\{q_k\}_{k \geq 1}$ of linear maps $q_k \in \text{Hom}^1(\odot^k(V[1]), V[1])$ such that the map

$$Q : \bigoplus_{n \geq 1} \odot^n V[1] \rightarrow \bigoplus_{n \geq 1} \odot^n V[1],$$

defined as

$$Q(v_1 \odot \dots \odot v_n) = \sum_{k=1}^n \sum_{\sigma \in S(k, n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \odot \dots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \dots \odot v_{\sigma(n)},$$

is a co-derivation on the graded co-algebra $\bigoplus_{n \geq 1} \odot^n V[1]$, i.e., $Q \circ Q = 0$.

Remark 6.3. Let $(L, d, [\cdot, \cdot])$ be a differential graded Lie algebra. Let q_1 be the suspension of the differential d , i.e.,

$$\begin{aligned} q_1 &:= d_{[1]} = \text{Id}_{\mathbb{C}[1]} \otimes d : V[1] \rightarrow V[1] \\ q_1(v_{[1]}) &= -(dv)_{[1]}. \end{aligned}$$

Then, define $q_2 \in \text{Hom}^1(\odot^2(V[1]), V[1])$ in the following way

$$q_2(v_{[1]} \odot w_{[1]}) = (-1)^{\deg v} [v, w]_{[1]}.$$

Finally, defining $q_k = 0$, for each $k \geq 3$, we endow the DGLA L with an L_∞ structure, i.e., *every DGLA is an L_∞ -algebra with zero higher multiplications.*

Example 6.4. Consider the DGLA $M[t, dt] = M \otimes \mathbb{C}[t, dt]$, where $\mathbb{C}[t, dt]$ is the differential graded algebra of polynomial differential forms over the affine line. For every $a \in \mathbb{C}$, define the *evaluation morphism* in the following way

$$\begin{aligned} e_a : M[t, dt] &\longrightarrow M, \\ e_a(\sum m_i t^i + n_i t^i dt) &= \sum m_i a^i. \end{aligned}$$

The evaluation morphism is a morphism of DGLAs which is a left inverse of the inclusion and it is a surjective quasi-isomorphism, for each a .

Next, define $K \subset L \times N \times M[t, dt] \times M[s, ds]$ as follows

$$K = \{(l, n, m_1(t, dt), m_2(s, ds)) \mid h(l) = e_1(m_2(s, ds)), g(n) = e_0(m_1(t, dt))\}.$$

K is a DGLA with bracket and differential defined as the natural ones on each component.

Consider the following morphisms of DGLAs:

$$e_1 : K \longrightarrow M, \quad (l, n, m_1(t, dt), m_2(s, ds)) \longmapsto e_1(m_1(t, dt))$$

and

$$e_0 : K \longrightarrow M, \quad (l, n, m_1(t, dt), m_2(s, ds)) \longmapsto e_0(m_2(s, ds)).$$

Let $H \subset K$ be defined as follow

$$H = \{k \in K \mid e_1(k) = e_0(k)\},$$

or written in detail

$$\begin{aligned} H = \{ & (l, n, m_1(t, dt), m_2(s, ds)) \in L \times N \times M[t, dt] \times M[s, ds] \mid \\ & h(l) = e_1(m_2(s, ds)), \ g(n) = e_0(m_1(t, dt)), \ e_1(m_1(t, dt)) = e_0(m_2(s, ds)) \}. \end{aligned}$$

For each $k = (l, n, m_1(t, dt), m_2(s, ds)) \in K$, the pair $m_1(t, dt)$ and $m_2(s, ds)$ has fixed values at one of the extremes of the unit interval. More precisely, the value of $m_1(t, dt)$ is fixed at the origin and $m_2(s, ds)$ is fixed at 1. If k also lies in H , then there are conditions on the other extremes: the value of $m_1(t, dt)$ at 1 has to coincide with the value of $m_2(s, ds)$ at 0.

Let

$$(8) \quad \begin{aligned} & H_{(h,g)} = \\ & \{(l, n, m(t, dt)) \in L \times N \times M[t, dt] \mid h(l) = e_1(m(t, dt)), \ g(n) = e_0(m(t, dt))\}. \end{aligned}$$

Since e_i are morphisms of DGLAs, it is clear that $H_{(h,g)}$ is a DGLA.

Moreover, considering the barycentric subdivision, we get an injective quasi-isomorphism

$$\begin{aligned} & H_{(h,g)} \hookrightarrow H, \\ & (l, n, m(t, dt)) \longmapsto (l, n, m(\frac{1}{2}t, dt), m(\frac{s+1}{2}, ds)). \end{aligned}$$

$H_{(h,g)}$ is the *differential graded Lie algebra associated with the pair (h, g) .*

Then, by Remark 6.3, an L_∞ structure on $H_{(h,g)}$ is defined by the following system of linear maps $q_k \in \text{Hom}^1(\odot^k(H_{(h,g)}[1], H_{(h,g)}[1]))$:

- $q_1(l, n, m(t, dt)) = (-dl, -dn, -dm(t, dt));$
- $q_2((l_1, n_1, m_1(t, dt)) \odot (l_2, n_2, m_2(t, dt))) =$
 $(-1)^{\deg_{H_{(h,g)}}(l_1, n_1, m_1(t, dt))}([l_1, l_2], [n_1, n_2], [m_1(t, dt), m_2(t, dt)]);$
- $q_k = 0$, for every $k \geq 3$.

A L_∞ -morphism $f_\infty : (V, q_1, q_2, q_3, \dots) \longrightarrow (W, p_1, p_2, p_3, \dots)$ of L_∞ -algebras is a sequence of degree zero linear maps

$$f_n : \bigodot^n V[1] \longrightarrow W[1], \quad n \geq 1,$$

such that the morphism of coalgebra

$$F : \bigoplus_{n \geq 1} \bigodot^n V[1] \longrightarrow \bigoplus_{n \geq 1} \bigodot^n W[1],$$

induced by $\sum_n f_n : \bigoplus_{n \geq 1} \bigodot^n V[1] \longrightarrow W[1]$, commutes with the codifferentials.

Sometimes this is the definition of a weak L_∞ -morphism; the strong (or linear) L_∞ -morphisms are the ones with $f_n = 0$, for each $n \geq 2$.

In particular, the *linear part* $f_1 : V[1] \longrightarrow W[1]$ of an L_∞ -morphism $f_\infty : (V, q_1, q_2, q_3, \dots) \longrightarrow (W, p_1, p_2, p_3, \dots)$ satisfies the condition $f_1 \circ q_1 = p_1 \circ f_1$, i.e., f_1 is a map of differential complexes $(V[1], q_1) \longrightarrow (W[1], p_1)$.

A *quasi-isomorphism* of L_∞ -algebra is an L_∞ -morphism, whose linear part is a quasi-isomorphism of differential complexes.

The key result in this setting is the *homotopical transfer of L_∞ structures*.

Theorem 6.5. *Let $(V, q_1, q_2, q_3, \dots)$ be an L_∞ -algebra and (C, δ) a differential graded vector space. If there exist two morphisms*

$$\pi : (V[1], q_1) \longrightarrow (C[1], \delta_{[1]}), \quad \iota : (C[1], \delta_{[1]}) \longrightarrow (V[1], q_1)$$

which are homotopy inverses, then there exists an L_∞ -algebra structure $(C, \langle \rangle_1, \langle \rangle_2, \dots)$ on C extending its differential complex structure, and making $(V, q_1, q_2, q_3, \dots)$ and $(C, \langle \rangle_1, \langle \rangle_2, \dots)$ be quasi-isomorphic L_∞ -algebra, via an L_∞ -quasi-isomorphism ι_∞ extending ι .

Proof. See [4, Theorem 4.1], [5] or [13]. □

Next, we use this theorem to transfer the L_∞ -structures of $H_{(h,g)}$, given by Example 6.4, to the cone $C_{(h,g)}$. We recall that $C_{(h,g)}^i = L^i \oplus N^i \oplus M^{i-1}$ and $D(l, n, m) = (dl, dn, -dm - g(n) + h(l))$.

Denote by $\langle \rangle_1 \in \text{Hom}^1(C_{(h,g)}[1], C_{(h,g)}[1])$ the suspended differential, i.e.,

$$\langle (l, n, m) \rangle_1 = (-dl, -dn, dm + g(n) - h(l)).$$

First of all, we note that we can define an integral operator \int_a^b on $M[t, dt]$, that is a linear map of degree -1 and extends the usual integration $\int_a^b : \mathbb{C}[t, dt] \longrightarrow \mathbb{C}$, i.e.,

$$\int_a^b : M[t, dt] \longrightarrow M \quad \int_a^b \left(\sum_i t^i m_i + t^i dt \cdot n_i \right) = \sum_i \left(\int_a^b t^i dt \right) n_i.$$

Next, define the following linear maps of degree 0:

$$\iota : C_{(h,g)}[1] \longrightarrow H_{(h,g)}[1]$$

$$\iota(l, n, m) = (l, n, (1-t)g(n) + th(l) + dtm)^a$$

and

$$\pi : H_{(h,g)}[1] \longrightarrow C_{(h,g)}[1]$$

$$\pi(l, n, m(t, dt)) = (l, n, \int_0^1 m(t, dt)).$$

Finally, define the homotopy $K \in \text{Hom}^{-1}(H_{(h,g)}[1], H_{(h,g)}[1])$ in the following way

$$K : H_{(h,g)}[1] \longrightarrow H_{(h,g)}[1]$$

$$K(l, n, m(t, dt)) = (0, 0, t \int_0^1 m(t, dt) - \int_0^t m(t, dt)).$$

Lemma 6.6. *ι and π are quasi-isomorphisms of complexes such that*

$$\pi \circ \iota = \text{id}_{C_{(h,g)}[1]} \quad \text{and} \quad \text{id}_{H_{(h,g)}[1]} - \iota \circ \pi = K \circ q_1 + q_1 \circ K$$

Proof. See [4, Lemma 3.2]. It is a straightforward computation. \square

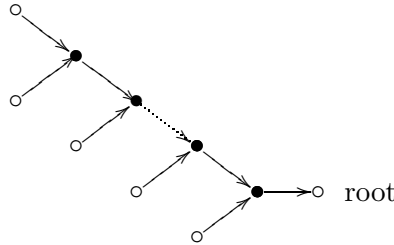
Thus, applying Theorem 6.5, we get an L_∞ -algebra structure $C_{(h,g)}$.

Corollary 6.7. *There exists an L_∞ -algebra structure $(C_{(h,g)}, \langle \rangle_1, \langle \rangle_2, \dots)$ on the complex $C_{(h,g)}$, that extends its differential structure, and it makes $(H_{(h,g)}, q_1, q_2, 0, \dots)$ and $(C_{(h,g)}, \langle \rangle_1, \langle \rangle_2, \dots)$ quasi-isomorphic L_∞ -algebras, via an L_∞ -quasi-isomorphism ι_∞ extending ι .*

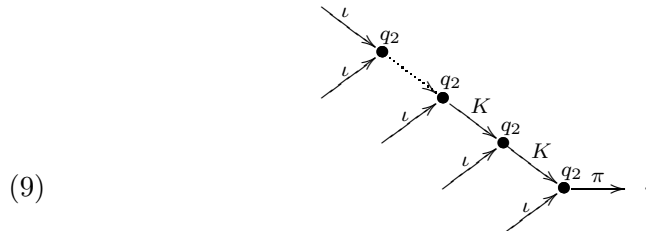
Remark 6.8. As explained in [3], [4], [5] and [13], we have an explicit description of the higher multiplication $\langle \rangle_n$ on $C_{(h,g)}$ in terms of rooted trees. Since

$$q_2(\text{Im } K \otimes \text{Im } K) \subseteq \ker \pi \cap \ker K \quad \text{and} \quad q_k = 0, \quad \forall k \geq 3,$$

it can be proved that we have to consider just the following rooted trees



decorated with the operators K, q_2, ι and π in the following way



^aIt is well defined, since $e_0(l, n, (1-t)g(n) + th(l) + dtm) = g(n)$ and $e_1(l, n, (1-t)g(n) + th(l) + dtm) = h(l)$.

Explicitly, for each $n \geq 2$, these diagrams give us the following formula for the higher multiplications $\langle \rangle_n$:

$$\begin{aligned} \langle \gamma_1 \odot \cdots \odot \gamma_n \rangle_n &= \\ &= \frac{(-1)^{n-2}}{2} \sum_{\sigma \in \Sigma_n} \varepsilon(\sigma) \pi q_2(\iota(\gamma_{\sigma(1)}) \odot K q_2(\iota(\gamma_{\sigma(2)}) \odot \cdots \odot K q_2(\iota(\gamma_{\sigma(n-1)}) \odot \iota(\gamma_{\sigma(n)}))) \cdots). \end{aligned}$$

The factor $1/2$ in the above formula accounts for the cardinality of the automorphisms group of the graph involved.

In particular, for each (l_1, n_1, m_1) and (l_2, n_2, m_2) in $C_{(h,g)}$ we have

$$\begin{aligned} \langle (l_1, n_1, m_1) \odot (l_2, n_2, m_2) \rangle_2 &= \\ \pi q_2(\iota(l_1, n_1, m_1) \odot \iota(l_2, n_2, m_2)) &= \\ (-1)^{\deg_{C_{(h,g)}}(l_1, n_1, m_1)} ([l_1, l_2], [n_1, n_2], & \\ ([g(n_1), m_2] + [m_1, g(n_2)]) \int_0^1 (1-t) dt + ([h(l_1), m_2] + [m_1, h(l_2)]) \int_0^1 t dt) &= \\ (-1)^{\deg_{C_{(h,g)}}(l_1, n_1, m_1)} & \\ ([l_1, l_2], [n_1, n_2], \frac{1}{2} ([g(n_1), m_2] + [m_1, g(n_2)] &+ [h(l_1), m_2] + [m_1, h(l_2)])) \rangle_2. \end{aligned}$$

If $(l_1, n_1, m_1) = (l_2, n_2, m_2) = (l, n, m) \in C_{(h,g)}^0[1]$, then

$$\langle (l, n, m)^{\odot 2} \rangle_2 = (-[l, l], -[n, n], -[m, g(n)] - [m, h(l)]).$$

Remark 6.9. All higher multiplications (for $n \geq 3$) vanish except the following ones:

$$(10) \quad \langle m_1 \odot \cdots \odot m_j \odot l \rangle_{j+1} \quad \text{and} \quad \langle m_1 \odot \cdots \odot m_j \odot n \rangle_{j+1}$$

or their linear combinations (for each $j \geq 2$). Here we use the notation $\gamma_i = m_i$ instead of $\gamma_i = (0, 0, m_i)$ and analogously for $\gamma_i = l$ or $\gamma_i = n$.

As in [4, Section 5], we can use Bernoulli's numbers to give an explicit description of the multiplications of (10). First of all, we recall that the Bernoulli's numbers B_n are defined by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160} + \cdots$$

Next, consider the multiplication

$$\begin{aligned} q_2(\iota(m) \odot \iota(l)) &= q_2((0, 0, dtm) \odot (l, 0, th(l))) \\ &= (-1)^{\deg_{H(h,g)}(0, 0, dtm)} (0, 0, tdt[m, h(l)]). \end{aligned}$$

Define recursively $\phi_j(t) \in \mathbb{Q}[t]$ and $I_j \in \mathbb{Q}$ as

$$\phi_1(t) = t, \quad I_j = \int_0^1 \phi_j(t) dt, \quad \phi_{j+1}(t) = \int_0^t \phi_j(s) ds - t I_j.$$

Then,

$$K((\phi_j(t) dt) m) = -\phi_{j+1}(t) m$$

and so

$$K q_2((dtm_1) \odot \phi_j(t) m_2) =$$

$$-(-1)^{\deg_M m_1 + 1} \phi_{j+1}(t)[m_1, m_2] = (-1)^{\deg_M m_1} \phi_{j+1}(t)[m_1, m_2].$$

Lemma 6.10. *The following formula holds*

$$\begin{aligned} & \langle m_1 \odot \cdots \odot m_j \odot l \rangle_{j+1} = \\ & (-1)^{n + \sum_{i=1}^j \deg_M(m_i)} I_j \sum_{\sigma \in \Sigma_j} \varepsilon(\sigma) [m_{\sigma(1)}, [m_{\sigma(2)}, \cdots [m_{\sigma(j)}, h(l)] \cdots]] \\ & = -(-1)^{\sum_{i=1}^j \deg_M(m_i)} \frac{B_j}{j!} \sum_{\sigma \in \Sigma_j} \varepsilon(\sigma) [m_{\sigma(1)}, [m_{\sigma(2)}, \cdots [m_{\sigma(j)}, h(l)] \cdots]]. \end{aligned}$$

Proof. See [4, Theorem 5.5]. □

In particular, if $m_{\sigma(i)} = m \in M^0$, for each i , then

$$\langle m^{\odot j} \odot l \rangle_{j+1} = -\frac{B_j}{j!} \sum_{\sigma \in \Sigma_j} [m, [m, \cdots [m, h(l)] \cdots]] = -B_j \operatorname{ad}_m^j(h(l)).$$

Next, consider the multiplication

$$\begin{aligned} q_2(\iota(m) \odot g(n)) &= q_2((0, 0, dtm) \odot (0, g(n), (1-t)g(n))) = \\ & (-1)^{\deg_{H(h,g)}(0,0,dtm)} (0, 0, (1-t)dt[m, g(n)]). \end{aligned}$$

In this case, define recursively $\varphi_j(t) \in \mathbb{Q}[t]$ as follows

$$\varphi_1(t) = 1 - t = 1 - \phi_1(t)$$

and

$$\varphi_{j+1}(t) = \int_0^t \varphi_j(s) ds - t \int_0^1 \varphi_j(s) ds.$$

We note that

$$\begin{aligned} \varphi_2(t) &= \int_0^t \varphi_1(s) ds - t \int_0^1 \varphi_1(s) ds = \\ &= \int_0^t (1 - \phi_1(s)) ds - t \int_0^1 (1 - \phi_1(s)) ds = \\ &= \left(\int_0^t ds - t \int_0^1 ds \right) - \left(\int_0^t \phi_1(s) ds - t \int_0^1 \phi_1(s) ds \right) = -\phi_2(t). \end{aligned}$$

Therefore, for $j \geq 2$ we get

$$\varphi_j(t) = -\phi_j(t).$$

Then,

$$K((\varphi_j(t) dt)m) = -\varphi_{j+1}(t)m$$

and so

$$\begin{aligned} & Kq_2((dtm_1) \odot \varphi_j(t)m_2) = \\ & -(-1)^{\deg_M m_1 + 1} \varphi_{j+1}(t)[m_1, m_2] = (-1)^{\deg_M m_1} \varphi_{j+1}(t)[m_1, m_2]. \end{aligned}$$

Lemma 6.11. *The following formula holds*

$$\begin{aligned} & \langle m_1 \odot \cdots \odot m_j \odot n \rangle_{j+1} = \\ & -(-1)^{n+\sum_{i=1}^j \deg_M(m_i)} I_j \sum_{\sigma \in S_j} \varepsilon(\sigma) [m_{\sigma(1)}, [m_{\sigma(2)}, \cdots [m_{\sigma(j)}, g(n)] \cdots]] \\ & = (-1)^{\sum_{i=1}^j \deg_M(m_i)} \frac{B_j}{j!} \sum_{\sigma \in S_j} \varepsilon(\sigma) [m_{\sigma(1)}, [m_{\sigma(2)}, \cdots [m_{\sigma(j)}, g(n)] \cdots]]. \end{aligned}$$

Proof. Analogous of Lemma 6.10. \square

In particular, if $m_{\sigma(i)} = m \in M^0$ for each i , then

$$\langle m^{\odot j} \odot n \rangle_{j+1} = \frac{B_j}{j!} \sum_{\sigma \in S_j} [m, [m, \cdots [m, g(n)] \cdots]] = B_j \operatorname{ad}_m^j(g(n)).$$

6.1. The Maurer-Cartan functor on the cone. Let T be an L_∞ -algebra. Then we can define (see [5],[13]) the *Maurer-Cartan functor* $\operatorname{MC}_T^\infty$ associated with T in the following way

$$\begin{aligned} & \operatorname{MC}_T^\infty : \mathbf{Art} \longrightarrow \mathbf{Set} \\ & \operatorname{MC}_T^\infty(A) = \left\{ \gamma \in T^1 \otimes m_A \mid \sum_{j \geq 1} \frac{\langle \gamma^{\odot j} \rangle_j}{j!} = 0 \right\}. \end{aligned}$$

Next, we want to prove that the Maurer-Cartan functor $\operatorname{MC}_{(h,g)}$ associated with the pair of morphisms of DGLAs $h : L \longrightarrow M$ and $g : N \longrightarrow M$ (introduced in Section 4.1) is exactly the Maurer-Cartan functor associated with the L_∞ structure on $C_{(h,g)}$ defined before.

By definition,

$$\begin{aligned} & \operatorname{MC}_{C_{(h,g)}}^\infty : \mathbf{Art} \longrightarrow \mathbf{Set} \\ & \operatorname{MC}_{C_{(h,g)}}^\infty(A) = \left\{ \gamma \in C_{(h,g)}[1]^0 \otimes m_A \mid \sum_{j \geq 1} \frac{\langle \gamma^{\odot j} \rangle_j}{j!} = 0 \right\}. \end{aligned}$$

Let $\gamma = (l, n, m) \in C_{(h,g)}[1]^0 \otimes m_A$, thus $l \in L^1 \otimes m_A$, $n \in N^1 \otimes m_A$ and $m \in M^0 \otimes m_A$. Then,

$$\langle (l, n, m) \rangle_1 = (-dl, -dn, dm + g(n) - h(l))$$

and

$$\langle (l, n, m)^{\odot 2} \rangle_2 = (-[l, l], -[n, n], -[m, g(n)] - [m, h(l)]).$$

Therefore, the Maurer-Cartan equation

$$\sum_{j \geq 1} \frac{\langle (l, n, m)^{\odot j} \rangle_j}{j!} = 0$$

splits into

$$dl + \frac{1}{2}[l, l] = dn + \frac{1}{2}[n, n] = 0$$

and

$$(11) \quad g(n) - h(l) + dm - \frac{1}{2}[m, g(n)] - \frac{1}{2}[m, h(l)] + \sum_{j \geq 3} \frac{\langle (l, n, m)^{\odot j} \rangle_j}{j!} = 0.$$

Since

$$\sum_{j \geq 3} \frac{\langle (l, n, m)^{\odot j} \rangle_j}{j!} = \sum_{j \geq 2} \frac{(j+1)}{(j+1)!} \langle m^{\odot j} \odot l \rangle_{j+1} + \sum_{j \geq 2} \frac{(j+1)}{(j+1)!} \langle m^{\odot j} \odot n \rangle_{j+1},$$

applying Lemma 6.10 and Lemma 6.11, Equation (11) becomes

$$\begin{aligned} & g(n) - h(l) + dm - \frac{1}{2}[m, g(n)] - \frac{1}{2}[m, h(l)] \\ & + \sum_{j \geq 2} \frac{B_j}{j!} \text{ad}_m^j(g(n)) - \sum_{j \geq 2} \frac{B_j}{j!} \text{ad}_m^j(h(l)) = 0. \end{aligned}$$

Since $B_0 = 1$ and $B_1 = -\frac{1}{2}$, we can write

$$0 = g(n) - h(l) + dm - [m, h(l)] + \sum_{j \geq 1} \frac{B_j}{j!} \text{ad}_m^j(g(n)) - \sum_{j \geq 1} \frac{B_j}{j!} \text{ad}_m^j(h(l))$$

and so

$$\begin{aligned} 0 &= dm - [m, h(l)] + \sum_{j \geq 0} \frac{B_j}{j!} \text{ad}_m^j(g(n)) - \sum_{j \geq 0} \frac{B_j}{j!} \text{ad}_m^j(h(l)) = \\ & dm - [m, h(l)] + \sum_{j \geq 0} \frac{B_j}{j!} \text{ad}_m^j(g(n) - h(l)). \end{aligned}$$

This implies that

$$0 = dm - [m, h(l)] + \frac{\text{ad}_m}{e^{\text{ad}_m} - id}(g(n) - h(l)).$$

Applying the operator $\frac{e^{\text{ad}_m} - id}{\text{ad}_m}$, we get

$$g(n) = h(l) + \frac{e^{\text{ad}_m} - id}{\text{ad}_m}([m, h(l)] - dm) = e^m * h(l).$$

In conclusion, the Maurer-Cartan equation for the L_∞ structure on $\mathcal{C}_{(h,g)}$ is

$$\begin{cases} dl + \frac{1}{2}[l, l] = 0 \\ dn + \frac{1}{2}[n, n] = 0 \\ e^m * h(l) = g(n). \end{cases}$$

Corollary 6.12. $\text{MC}_{(h,g)} \cong \text{MC}_{\mathcal{C}_{(h,g)}}^\infty$.

Given an L_∞ -algebra T , two elements x and $y \in \text{MC}_T^\infty(A)$ are *homotopy equivalent* if there exists $g[s, ds] \in \text{MC}_{T[s, ds]}^\infty(A)$ with $g(0) = x$ and $g(1) = y$. Then, the *deformation functor* Def_T^∞ associated with T is $\text{MC}_T^\infty / \text{homotopy}$. Moreover, if two L_∞ -algebras are quasi-isomorphic, then there exists an isomorphism between the associated deformation functors [5]. In particular, Corollary 6.7 implies the following result.

Corollary 6.13. $\text{Def}_{\mathcal{C}_{(h,g)}}^\infty \cong \text{Def}_{H_{(h,g)}}$.

Next, we prove that there is also an isomorphism between the functor $\text{Def}_{(h,g)}$ and the deformation functor $\text{Def}_{C(h,g)}^\infty$. First of all, we state two useful lemmas. We will use the notation $e_a(p(t, dt)) := p(a)$, for each $p(t, dt) \in M[t, dt]$ and $a \in \mathbb{C}$.

Lemma 6.14. *Let M be a differential graded Lie algebra and let $A \in \mathbf{Art}$. Then, for any x in $\text{MC}_M(A)$ and any $g(t) \in M^0[t] \otimes m_A$, with $g(0) = 0$, the element $e^{g(t)} * x$ is an element of $\text{MC}_{M[t, dt]}(A)$. Moreover all the elements of $\text{MC}_{M[t, dt]}(A)$ are obtained in this way.*

Proof. See [4, Corollary 7.2]. \square

Lemma 6.15. *Let $x(t, dt) \in \text{MC}_{M[t, dt]}(A)$, $\mu(t, dt) \in M[t, dt]^0 \otimes m_A$, such that $\mu(0) = 0$, and*

$$e^{\mu(t, dt)} * x(t, dt) = x(t, dt).$$

Then, $e^{\mu(1)} \in \text{Stab}_A(x(1))$, i.e., there exists $C \in M^{-1}$ such that $\mu(1) = d_M C + [x(1), C]$, where d_M is the differential in M .

Remark 6.16. The proof of the case $x(t, dt) = 0$ is contained in [4, Theorem 7.4].

Proof. First, suppose that $x(t, dt) = x \in M$. Thus, we have $e^{\mu(t, dt)} * x = x$ and so

$$(12) \quad d(\mu(t, dt)) + [x, \mu(t, dt)] = 0 \in (M[t, dt])^1 \otimes m_A.$$

If we write $\mu(t, dt) = \mu^0(t) + \mu^{-1}(t)dt$, with $\mu^0(t) \in M[t]^0$ and $\mu^{-1}(t) \in M[t]^{-1}$, then Equation (12) becomes

$$\dot{\mu}^0(t)dt + d_M \mu^0(t) + d_M \mu^{-1}(t)dt + [x, \mu^0(t)] + [x, \mu^{-1}(t)]dt = 0,$$

or, equivalently,

$$\begin{cases} \dot{\mu}^0 + d_M \mu^{-1}(t) + [x, \mu^{-1}(t)] = 0 \\ d_M \mu^0(t) + [x, \mu^0(t)] = 0. \end{cases}$$

Thus, for any fixed $\mu^{-1}(t)$, we get

$$\begin{aligned} \mu^0(t) &= \\ &- \int_0^t d_M \mu^{-1}(s)ds - \int_0^t [x, \mu^{-1}(s)]ds = -d_M \int_0^t \mu^{-1}(s)ds - [x, \int_0^t \mu^{-1}(s)ds]. \end{aligned}$$

Let $C = - \int_0^1 \mu^{-1}(s)ds \in M^{-1}$. Therefore, $\mu(1) = \mu^0(1) = d_M C + [x, C]$ or, analogously, $e^{\mu(1)} \in \text{Stab}_A(x)$. This concludes the proof in the case $x(t, dt) = x$.

Next, consider the general case of a Maurer-Cartan element $x(t, dt) \in \text{MC}_{M[t, dt]}(A)$. Lemma 6.14 implies the existence of $g(t) \in M^0[t]$ such that $g(0) = 0$ and

$$x(t, dt) = e^{g(t)} * x(0).$$

Therefore, the hypothesis $e^{\mu(t, dt)} * x(t, dt) = x(t, dt)$, implies

$$e^{-g(t)} e^{\mu(t, dt)} e^{g(t)} * x(0) = x(0).$$

Let $q(t, dt) = -g(t) \bullet \mu(t, dt) \bullet g(t)$. If we write $\mu(t, dt) = \mu^0(t) + \mu^{-1}(t)dt$ and $q(t, dt) = q^0(t) + q^{-1}(t)dt$, then $q^0(t) = -g(t) \bullet \mu^0(t) \bullet g(t)$.

By the previous consideration applied to $e^{q(t,dt)} * x_0 = x_0$, we conclude that $e^{q(1)} \in \text{Stab}_A(x(0))$.

The main property of irrelevant stabilizer asserts that

$$\forall a \in M^0 \otimes A \quad e^a \text{Stab}_A(x) e^{-a} = \text{Stab}_A(y), \quad \text{with} \quad y = e^a * x.$$

Therefore, $e^{\mu(1)} = e^{g(1)} e^{q(1)} e^{-g(1)} \in \text{Stab}_A(y)$, with $y = e^{g(1)} * x(0) = x(1)$.

Equivalently, there exists $C \in M^{-1}$ such that $\mu(1) = d_M C + [x(1), C]$. \square

Theorem 6.17.

$$\text{Def}_{C(h,g)}^\infty = \frac{\text{MC}_{C(h,g)}^\infty}{\text{homotopy}} \simeq \frac{\text{MC}_{(h,g)}}{\text{gauge}} = \text{Def}_{(h,g)}.$$

Proof. This theorem is a generalization of [4, Theorem 7.4].

First, we show that *gauge equivalence implies homotopy equivalence*. Let (l_0, n_0, m_0) and (l_1, n_1, m_1) in $\text{MC}_{(h,g)}(A)$, for some $A \in \mathbf{Art}$; in particular, $e^{m_0} * h(l_0) = g(n_0)$ and $e^{m_1} * h(l_1) = g(n_1)$.

Suppose that they are gauge equivalent elements, i.e., there exist $a \in L^0 \otimes m_A$, $b \in N^0 \otimes m_A$ and $c \in M^{-1} \otimes m_A$ such that

$$l_1 = e^a * l_0, \quad n_1 = e^b * n_0, \quad m_1 = g(b) \bullet T \bullet m_0 \bullet (-h(a)),$$

with $T = dc + [g(n_0), c]$ (and so $e^T \in \text{Stab}_A(g(n_0))$).

Let $\tilde{l}(s, ds) = e^{sa} * l_0 \in L[s, ds] \otimes m_A$, $\tilde{n}(s, ds) = e^{sb} * n_0 \in N[s, ds] \otimes m_A$ and $T(s) = d(sc) + [g(n_0), sc]$. By Lemma 6.14, \tilde{l} and \tilde{n} satisfy the Maurer-Cartan equation and $h(\tilde{l}) = e^{h(sa)} * h(l_0)$ and $g(\tilde{n}) = e^{g(sb)} * g(n_0)$.

Define $\tilde{m} = g(sb) \bullet T(s) \bullet m_0 \bullet (-h(sa))$; then,

$$\begin{aligned} e^{\tilde{m}} * h(\tilde{l}) &= e^{g(sb) \bullet T(s) \bullet m_0 \bullet (-h(sa))} * h(\tilde{l}) = \\ &= e^{g(sb) \bullet T(s) \bullet m_0} * h(l_0) = e^{g(sb) \bullet T(s)} * g(n_0) = g(\tilde{n}). \end{aligned}$$

Therefore, $(\tilde{l}, \tilde{n}, \tilde{m}) \in \text{MC}_{C(h,g)[t,dt]}^\infty(A)$. Moreover, $\tilde{l}(0) = l_0$, $\tilde{l}(1) = l_1$, $\tilde{n}(0) = n_0$, $\tilde{n}(1) = n_1$, $\tilde{m}(0) = m_0$ and $\tilde{m}(1) = g(b) \bullet T \bullet m_0 \bullet (-h(a)) = m_1$, i.e., (l_0, n_0, m_0) and (l_1, n_1, m_1) are homotopy equivalent.

Next, we show that *homotopy equivalence implies gauge equivalence*. Let (l_0, n_0, m_0) and (l_1, n_1, m_1) be homotopy equivalent elements in $\text{MC}_{C(h,g)}^\infty(A)$.

Thus, there exists $(\tilde{l}, \tilde{n}, \tilde{m}) \in \text{MC}_{C(h,g)[t,dt]}^\infty(A)$ such that

$$d\tilde{l} + \frac{1}{2}[\tilde{l}, \tilde{l}] = 0, \quad d\tilde{n} + \frac{1}{2}[\tilde{n}, \tilde{n}] = 0, \quad g(\tilde{n}) = e^{\tilde{m}} * h(\tilde{l})$$

and

$$\begin{cases} (\tilde{l}(0), \tilde{n}(0), \tilde{m}(0)) = (l_0, n_0, m_0) \\ (\tilde{l}(1), \tilde{n}(1), \tilde{m}(1)) = (l_1, n_1, m_1). \end{cases}$$

In particular, \tilde{l} and \tilde{n} satisfy the Maurer-Cartan equation in $L[t, dt]$ and $N[t, dt]$, respectively. Applying Lemma 6.14, there exist degree zero elements $\lambda(t) \in L[t] \otimes m_A$ and $\nu(t) \in N[t] \otimes m_A$, such that $\lambda(0) = 0$, $\tilde{l} = e^\lambda * l_0$, $\nu(0) = 0$ and $\tilde{n} = e^\nu * n_0$.

This implies that $h(\tilde{l}) = e^{h(\lambda)} * h(l_0)$, $g(\tilde{n}) = e^{g(\nu)} * g(n_0)$ and, for $s = 1$, that

$$l_1 = e^{\lambda(1)} * l_0 \quad \text{and} \quad n_1 = e^{\nu(1)} * n_0.$$

Moreover, we note that

$$e^{g(\nu) \bullet m_0 \bullet (-h(\lambda))} * h(\tilde{l}) = e^{g(\nu) \bullet m_0} * h(l_0) = e^{g(\nu)} * g(n_0) = g(\tilde{n}).$$

Let $\mu = \tilde{m} \bullet h(\lambda) \bullet (-m_0) \bullet (-g(\nu)) \in M^0[t, dt] \otimes m_A$ so that $\tilde{m} = \mu \bullet g(\nu) \bullet m_0 \bullet (-h(\lambda))$. Then $\mu(0) = 0$ and $e^\mu * g(\tilde{n}) = g(\tilde{n})$.

Therefore, by Lemma 6.15, there exists $C \in M^{-1}$, such that $\mu(1) = d_M(C) + [g(n_1), C]$ and so $e^{\mu(1)} \in \text{Stab}_A(g(n_1))$. Then, $m_1 = \tilde{m}(1) = \mu(1) \bullet g(\nu(1)) \bullet m_0 \bullet (-h(\lambda_1))$. Applying the main property of the irrelevant stabilizers, there exists $C' \in M^{-1}$ such that

$$\mu(1) \bullet g(\nu(1)) = g(\nu(1)) \bullet T',$$

with $T' = dC' + [g(n_0), C']$ and $e^{T'} \in \text{Stab}_A(g(n_0))$. Thus, $m_1 = g(\nu(1)) \bullet T' \bullet m_0 \bullet (-h(\lambda_1))$.

In conclusion, if (l_0, n_0, m_0) and (l_1, n_1, m_1) are homotopy equivalent, then there exists $(\lambda(1), \nu(1)) \in (L^0 \otimes m_A) \times (N^0 \otimes m_A)$ and $T' = dC' + [g(n_0), C']$, for some $C' \in M^{-1}$, such that

$$\begin{cases} l_1 = e^{\lambda(1)} * l_0 \\ n_1 = e^{\nu(1)} * n_0 \\ m_1 = g(\nu(1)) \bullet T' \bullet m_0 \bullet (-h(\lambda(1))), \end{cases}$$

i.e., (l_0, n_0, m_0) and (l_1, n_1, m_1) are gauge equivalent. \square

Corollary 6.18. $\text{Def}_{(h,g)} \cong \text{Def}_{H_{(h,g)}}$.

Therefore, by suitably choosing L, M , and h, g , we have an explicit description of the DGLA that controls the infinitesimal deformations of holomorphic maps.

Theorem 6.19. *Let $f : X \rightarrow Y$ be a holomorphic map. Then, the DGLA $H_{(h,g)}$ associated with the morphisms $h : L \hookrightarrow KS_{X \times Y}$ and $g = (p^*, q^*) : KS_X \times KS_Y \rightarrow KS_{X \times Y}$ (introduced in Section 4) controls infinitesimal deformations of f , i.e.,*

$$\text{Def}_{H_{(h,g)}} \cong \text{Def}(f).$$

Proof. It is sufficient to apply Theorem 5.11 and Corollary 6.18. \square

REFERENCES

- [1] M. Artin, *Deformations of Singularities*, Tata Institute of Fundamental Research, Bombay, (1976).
- [2] F. Catanese, *Moduli of algebraic surfaces*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Mathematics, **1337**, Springer-Verlag, New York/Berlin, (1988), 1-83.
- [3] D. Fiorenza, *Sums over graphs and integration over discrete groupoids*, Applied Categorical Structures, **14** (No. 4), (2006), 313-350; [arXiv:math.CT/0211389](https://arxiv.org/abs/math.CT/0211389).
- [4] D. Fiorenza, M. Manetti, *L_∞ structures on mapping cones*; Algebra & Number Theory, **1**, (2007), 301-330; [arXiv:math.QA/0601312v3](https://arxiv.org/abs/math.QA/0601312v3).
- [5] K. Fukaya, *Deformation theory, homological algebra and mirror symmetry*, Geometry and physics of branes (Como, 2001), Ser. High Energy Phys. Cosmol. Gravit., IOP Bristol, (2003), 121-209. Electronic version available at <http://www.math.kyoto-u.ac.jp/%7Efukaya/como.dvi>
- [6] W.M. Goldman, J.J. Millson, *The homotopy invariance of the Kuranishi space*, Ill. J. Math., **34**, (1990), 337-367.

- [7] A. Grothendieck, *Technique de Descente et théorèmes d'existence en géométrie algébrique. II. Le théorèmes d'existence en théorie formelle des modules*, Séminaire Bourbaki, t. 12, Exp. no. **195**, (1959-1960).
- [8] E. Horikawa, *On deformations of holomorphic maps I, II*, J. Math. Soc. Japan, **25** (No.3), (1973), 372-396; **26** (No.4), (1974), 647-667.
- [9] E. Horikawa, *On deformations of holomorphic maps III*, Math. Annalen, **222**, (1976), 275-282.
- [10] D. Iacono, *Differential Graded Lie Algebras and deformations of Holomorphic Maps*, PhD Thesis, Roma, (2006), [arXiv:math.AG/0701091](#)
- [11] D. Iacono, *A semiregularity map annihilating obstructions to deforming holomorphic maps*, Preprint [arXiv:0707.2454v1](#).
- [12] K. Kodaira, D.C. Spencer, *On Deformations of Complex Analytic Structures, II* Ann. of Math., **67** (2), (1958), 403-466.
- [13] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*. Letters in Mathematical Physics, **66**, (2003), 157-216; [arXiv:q-alg/9709040](#).
- [14] M. Kontsevich, *Topics in algebra-deformation theory*, unpublished notes on a course given at the University of Berkley, (1994).
- [15] M. Manetti, *Lectures on deformations of complex manifolds*, Rend. Mat. Appl. (7), **24**, (2004), 1-183; [arXiv:math.AG/0507286](#).
- [16] M. Manetti, *Lie description of higher obstructions to deforming submanifolds*, Preprint [arXiv:math.AG/0507287 v2](#) 7 Oct 2005.
- [17] M. Namba, *Families of meromorphic functions on compact Riemann surfaces*, Lecture Notes in Mathematics, **767**, Springer-Verlag, New York/Berlin, (1979).
- [18] Z. Ran, *Deformations of maps*, Algebraic curves and projective geometry (Trento 1989), Lecture Notes in Mathematics, **1389**, Springer-Verlag, New York/Berlin, (1989), 246-253.
- [19] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc., **130**, (1968), 208-222.
- [20] E. Sernesi, *Deformations of Algebraic Schemes*, Grundlehren der mathematischen Wissenschaften, **334**, Springer-Verlag, New York/Berlin, 2006.

SISSA-ISAS,

VIA BEIRUT, 2-4, 34014 TRIESTE, ITALY.

E-mail address: iacono@sisssa.it